

The curvature of branes, currents and gravity in matrix models

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Abstract

The curvature of brane solutions in Yang-Mills matrix models is expressed in terms of conserved currents associated with global symmetries of the model. This implies a relation between the Ricci tensor and the energy-momentum tensor due to the basic matrix model action, without invoking an Einstein-Hilbert term. The coupling is governed by the extrinsic curvature of the brane embedding, which arises naturally for compactified brane solutions. The effective gravity on the brane is thereby related to the compactification moduli, and protected from quantum corrections due to the relation with global symmetries.

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1 Introduction

Matrix models provide remarkable candidates for a pre-geometric theory of fundamental interactions including gravity. In particular, the IKKT model [1] was proposed originally as a non-perturbative definition of IIB string theory, and the BFSS model ([2], cf. [3]) as a non-perturbative definition of M-theory. It is well-known that the models admit indeed brane solutions, consistent with IIB supergravity resp. 11-dimensional supergravity. However they clearly go beyond supergravity, and should provide a non-perturbative quantum theory by integrating over the space of matrices. In particular the IKKT model is well suited for numerical simulations, and evidence was reported recently [4] for the emergence of 3+1-dimensional space-time. Therefore a better theoretical understanding of the dynamics of branes and their geometry and gravity is very important.

A systematic study of the effective geometry of brane solutions in the matrix model was undertaken in recent years [5–8]. This led to a description of branes as quantized symplectic

submanifold embedded in $\mathbb{R}^{9,1}$, with effective metric $G^{ab} \sim \theta^{aa'} \theta^{bb'} g_{a'b'}$ determined by the Poisson structure and the embedding metric g . The dynamical metric G^{ab} governs all matter and fields on the branes, and must therefore be interpreted as gravitational metric. The relation with string theory or supergravity is seen by relating θ with the B field, g with the closed string (bulk) metric, and G with the open string metric on the brane [10].

Since the basic solutions of the model are branes with a dynamical metric, one is led to a picture of brane-worlds. The mechanism for gravity is however not obvious, and there are several possible scenarios. One possibility is that quantization leads to an induced gravity action, which is however delicate and leads to fine-tuning issues. Another mechanism¹ is holography, and indeed the bulk metric of supergravity seems to arise quantum mechanically from the brane description [1, 11–15]. However, to obtain an acceptable 4-dimensional gravity the 10-dimensional bulk must be compactified in this scenario, leading to a landscape of vacua with its inherent lack of predictivity [16].

On the other hand, these conventional pictures miss the basic fact that the metric is not a fundamental degree of freedom in the matrix model, but a derived quantity. This means that the geometrical equations of motion admit solutions of the brane embedding and its Poisson structure given by harmonic brane embeddings and excitations of θ^{ab} . This suggests a different, “emergent” gravity mechanism, based on the basic matrix model action rather than quantum effects. Indeed excitations of the Poisson structure lead to Ricci-flat perturbations [17] on flat \mathbb{R}^4 and certain self-dual geometries [18]. However on flat branes, matter does not seem to induce the metric perturbations required for gravity. Remarkably, it *does* on branes with non-trivial extrinsic curvature as pointed out in [19], and indeed Newtonian gravity arises within harmonic brane excitations. A similar mechanism is realized on compactified brane solutions $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^{10}$, where the extrinsic curvature arises from the compactification, whose moduli become 4-dimensional gravitational modes [20]. This mechanism is very interesting, but its analysis was limited to the linearized regime, and obscured by certain “mixing” terms whose significance remained unclear.

In this paper, we establish new techniques and results which allow to efficiently compute the curvature of branes in the IKKT model. This provides significant new insights into the above mechanism for emergent gravity. We first establish a description of the geometry in terms of an over-complete frame, based on the currents associated with the global $SO(D)$ symmetry of the model. The curvature can then be computed using techniques from projective modules. We obtain an explicit and compact expression for a broad class of geometries including generalized almost-Kähler geometry, adapted to the case of Minkowski signature. This class of geometries is argued to be sufficiently general and dynamically preferred by the model. The currents are useful because their conservation law encodes the equations of motion of the brane, and moreover the energy-momentum tensor of matter acts as source for currents. Using the results for the curvature, it follows that the energy-momentum tensor couples indeed to the Ricci tensor, albeit in an indirect way mediated by a tensor \mathcal{P} which also provides an additional vacuum contribution. This coupling \mathcal{P} depends on the extrinsic curvature of the brane as well as the Poisson tensor, hence on the brane compactification. Assuming that \mathcal{P} respects the effective 4-dimensional Lorentz symmetry, the Einstein equations should be recovered, up to vacuum contributions. Although more work is required to clarify \mathcal{P} and

¹A rather different gravitational interpretation of the IKKT equations of motion was proposed in [21], whose significance for the brane solutions is not clear.

its dynamics, this provides strong support for the emergent gravity scenario on compactified branes $\mathcal{M} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$ in matrix models.

The relation with global symmetries and with non-commutative gauge theory make this mechanism for gravity very attractive for quantization, for the maximally supersymmetric IKKT model. Since a compactification on fuzzy extra dimensions \mathcal{K}_N can be viewed as non-trivial vacuum in a $U(N)$ noncommutative $\mathcal{N} = 4$ SYM theory, the model is expected to be UV finite on such backgrounds. This is no longer the case for more than 4 noncompact dimensions, which may explain the emergence of 3+1 non-compact dimensions as observed numerically in [4]. Furthermore, since the currents are associated with global symmetries of the model, the gravitational degrees of freedom can be expected to remain massless at the quantum level, and the mechanism should be protected from fine-tuning problems.

This paper is organized as follows. After reviewing the description of noncommutative branes in matrix models, we introduce the generalized frame formalism for the embedding geometry in section 3.2 based on the currents. This is extended to the effective geometry in section 3.4, where a special class of geometries is introduced. The Ricci tensor for the effective geometry is then computed in section 3.6. We also discuss the equation of motion for the Poisson structure via energy-momentum conservation in section 3.7, and include a section on flux stabilization. The inhomogeneous current conservation law is derived in section 4, and some technical details are elaborated in the appendices.

2 Matrix models and their geometry

We briefly collect the essential ingredients of the matrix model framework and its effective geometry, referring to the review [8] for more details.

2.1 The IKKT model

The starting point is given by a matrix model of Yang-Mills type,

$$S = -\frac{\Lambda_0^4}{4} \text{Tr} ([X^A, X^B][X^C, X^D]\eta_{AC}\eta_{BD} + 2\bar{\Psi}\gamma_A[X^A, \Psi]) = S_{\text{YM}} + S_{\Psi} \quad (2.1)$$

where the X^A are Hermitian matrices, i.e. operators acting on a separable Hilbert space \mathcal{H} . The indices of the matrices run from 0 to $D - 1$, and will be raised or lowered with the invariant tensor η_{AB} of $SO(D - 1, 1)$. We also introduce a parameter Λ_0 of dimension $[L]^{-1}$, so that the X^A have dimension length. We focus on the maximally supersymmetric IKKT or IIB model [1] with $D = 10$, which is best suited for quantization. It is obtained from the $\mathcal{N} = 1$ $U(N)$ SYM on \mathbb{R}^{10} dimensionally reduced to a point, and taking $N \rightarrow \infty$. Then Ψ is a matrix-valued Majorana Weyl spinor of $SO(9, 1)$. The model enjoys the fundamental gauge symmetry

$$X^A \rightarrow U^{-1}X^AU, \quad \Psi \rightarrow U^{-1}\Psi U, \quad U \in U(\mathcal{H}) \quad (2.2)$$

as well as the 10-dimensional Poincaré symmetry

$$\begin{aligned} X^A &\rightarrow \Lambda(g)^A_B X^B, & \Psi_\alpha &\rightarrow \tilde{\pi}(g)_\alpha^\beta \Psi_\beta, & g &\in \widetilde{SO}(9, 1), \\ X^A &\rightarrow X^A + c^A \mathbf{1}, & & & c^A &\in \mathbb{R}^{10} \end{aligned} \quad (2.3)$$

and a $\mathcal{N} = 2$ matrix supersymmetry [1]. The tilde indicates the corresponding spin group. We define the matrix Laplacian as

$$\square\Phi := [X_B, [X^B, \Phi]] \quad (2.4)$$

for any matrix $\Phi \in \mathcal{L}(\mathcal{H})$. Then the equations of motion of the model take the following form

$$\square X^A = [X_B, [X^B, X^A]] = 0, \quad (2.5)$$

assuming $\Psi = 0$. Analogous statements hold more generally to matrix models of Yang-Mills type, with Euclidean or Minkowski signature.

2.2 Noncommutative branes and their geometry

Now we focus on matrix configurations which describe embedded noncommutative (NC) branes. This means that the X^A can be interpreted as quantized embedding functions [8]

$$X^A \sim x^A : \quad \mathcal{M}^{2n} \hookrightarrow \mathbb{R}^{10} \quad (2.6)$$

of a $2n$ -dimensional submanifold of \mathbb{R}^{10} . More precisely, there should be some quantization map $\mathcal{I} : \mathcal{C}(\mathcal{M}) \rightarrow \mathcal{A} \subset L(\mathcal{H})$ which maps classical functions on \mathcal{M} to a noncommutative (matrix) algebra of functions, such that commutators can be interpreted as quantized Poisson brackets. In the semi-classical limit indicated by \sim , matrices are identified with functions via \mathcal{I} , and commutators are replaced by Poisson brackets; for a more extensive introduction see e.g. [8, 9]. One can then locally choose $2n$ independent coordinate functions x^a , $a = 1, \dots, 2n$ among the x^A , and their commutators

$$[X^a, X^b] \sim i\{x^a, x^b\} = i\theta^{ab}(x) \quad (2.7)$$

encode a quantized Poisson structure on $(\mathcal{M}^{2n}, \theta^{ab})$. These θ^{ab} have dimension $[L^2]$ and set a typical scale of noncommutativity Λ_{NC}^{-2} . We will assume that θ^{ab} is non-degenerate², so that the inverse matrix θ_{ab}^{-1} defines a symplectic form on $\mathcal{M}^{2n} \subset \mathbb{R}^{10}$. This submanifold is equipped with the induced metric

$$g_{ab}(x) = \partial_a x^A \partial_b x_A \quad (2.8)$$

which is the pull-back of η_{AB} . However, this is *not* the effective metric on \mathcal{M} . To understand the effective metric and gravity, we need to consider matter on the brane \mathcal{M} . Bosonic matter or fields arise from nonabelian fluctuations of the matrices around a stack $X^A \otimes \mathbb{1}_n$ of coinciding branes, while fermionic matter arises from Ψ in (2.1). It turns out that in the semi-classical limit, the effective action for such fields is governed by a universal effective metric G^{ab} . It can be obtained most easily by considering the action of an additional scalar field ϕ coupled to the matrix model in a gauge-invariant way, with action

$$\begin{aligned} S[\phi] &= -\frac{\Lambda_0^4}{2} \text{Tr} [X_A, \phi] [X^A, \phi] \sim \frac{\Lambda_0^4}{2(2\pi)^n} \int d^{2n}x \sqrt{|\theta^{-1}|} \theta^{aa'} \theta^{bb'} g_{a'b'} \partial_a \phi \partial_b \phi \\ &= \frac{\Lambda_0^4}{2(2\pi)^n} \int d^{2n}x \sqrt{|G|} G^{ab} \partial_a \phi \partial_b \phi. \end{aligned} \quad (2.9)$$

²If the Poisson structure is degenerate, then the fluctuations propagate only along the symplectic leaves.

Therefore the effective metric is given by [6]

$$\begin{aligned} G^{ab} &= e^{-\sigma} \gamma^{ab}, & \gamma^{ab} &= \theta^{aa'} \theta^{bb'} g_{a'b'} \\ e^{-\sigma} &= \left(\frac{\det \theta_{ab}^{-1}}{\det G_{ab}} \right)^{\frac{1}{2}} = \left(\frac{\det \theta_{ab}^{-1}}{\det g_{ab}} \right)^{\frac{1}{2(n-1)}}. \end{aligned} \quad (2.10)$$

It is useful to consider the conformally equivalent metric³ γ^{ab} which satisfies

$$\sqrt{|\theta^{-1}|} \gamma^{ab} = \sqrt{|G|} G^{ab}. \quad (2.11)$$

The effective metric G^{ab} is encoded in the matrix Laplace operator, which can be seen from the following result [8] for the semi-classical limit

$$\square \Phi = [X_A, [X^A, \Phi]] \sim -e^\sigma \square_G \phi \quad (2.12)$$

acting on scalar fields $\Phi \sim \phi$. In particular, the matrix equations of motion (2.5) take the simple form $\square X^A \sim -e^\sigma \square_G x^A = 0$. This means that the embedding functions $x^A \sim X^A$ are harmonic functions with respect to G . Furthermore, the bosonic part of the matrix model action (2.1) can be written in the semi-classical limit as follows

$$S_{\text{YM}} \sim \frac{\Lambda_0^4}{4(2\pi)^{2n}} \int d^{2n}x \sqrt{|\theta^{-1}|} \gamma^{ab} g_{ab}. \quad (2.13)$$

Compactified brane solutions. Of particular interest here are branes with compactified extra dimensions

$$\mathcal{M}^{2n} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D \quad (2.14)$$

where the extrinsic curvature is predominantly due to $\mathcal{K} \subset \mathbb{R}^D$, while the embedding of \mathcal{M}^4 is approximately flat. Such solutions including $\mathcal{K} = T^2$ and $\mathcal{K} = S^3 \times S^1$ have been given recently [22], where \mathcal{K} is rotating along \mathcal{M}^4 and stabilized by angular momentum. This is possible because of "split noncommutativity", where the Poisson structure relates the compact space \mathcal{M}^4 with the non-compact space \mathcal{K} ,

$$\theta^{\mu i} = \{x^\mu, y^i\} \neq 0. \quad (2.15)$$

Here x^μ are coordinates on \mathcal{M}^4 and y^i are coordinates on \mathcal{K} . As pointed out in [20], such a structure relates the perturbations of \mathcal{K} to perturbations of the effective metric on \mathcal{M}^4 , and thereby links the Ricci tensor to the energy-momentum tensor. This leads to a novel mechanism for 4-dimensional gravity. The aim of this paper is to understand better this mechanism, by computing the intrinsic curvature on the brane in the presence of matter.

3 Currents and geometry

To compute the curvature of \mathcal{M} directly from the metric γ_{ab} is complicated and not illuminating. We will develop a suitable generalized frame formalism, which allows to express the curvature efficiently in terms of the $SO(D)$ conserved currents. This will be the key to a better understanding of the effective gravity on the branes.

³More abstractly, this can be stated as $(\alpha, \beta)_\gamma = (i_\alpha \theta, i_\beta \theta)_g$ where $\theta = \frac{1}{2} \theta^{ab} \partial_a \wedge \partial_b$.

3.1 Currents and conservation laws

The matrix model (2.1) is invariant under the $SO(D)$ resp. $SO(1, D-1)$ symmetries⁴

$$\delta X^A = (\lambda^\alpha)_B^A X^B \quad (3.1)$$

for $\lambda^\alpha \in \mathfrak{so}(D)$. Setting $\Psi = 0$ for now, they lead to conserved currents in complete analogy to quantum field theory,

$$[X^A, \tilde{J}_A^\alpha] = 0, \quad \tilde{J}_A^\alpha = \frac{1}{2} \lambda_{CD}^\alpha \{X^C, [X_A, X^D]\} \quad (3.2)$$

with anti-symmetric $\lambda_{AB}^\alpha \in \mathfrak{so}(D)$. This can be verified directly using the equation of motion (2.5), or more conceptually via a matrix version of the Noether theorem, as elaborated in appendix A. In the semi-classical limit, this reduces to

$$\begin{aligned} \nabla^a J_a^\alpha &= 0 & J_a^\alpha &= x^A \lambda_{AB}^\alpha \partial_a x^B \\ \tilde{J}_A^\alpha &\sim i \theta^{ab} \partial_a x_A J_b^\alpha, \end{aligned} \quad (3.3)$$

where ∇ is the Levi-Civita connection corresponding to the effective metric G . The conservation law in the presence of matter will be discussed in section 4.1. These conservation laws completely capture the equation of motion for the modes which preserve $S^{D-1} \subset \mathbb{R}^D$. Due to their origin from global symmetries, these conservation laws are expected to be protected from quantum corrections, as usual in quantum field theory. This makes them well suited to describe the geometry of the model and its dynamics.

3.2 Generalized embedding frame

The above currents are naturally viewed as one-forms in the cotangent bundle $T^*\mathcal{M}$,

$$J^\alpha = x^A \lambda_{AB}^\alpha dx^B = J_a^\alpha d\xi^a, \quad J_a^\alpha = x \lambda^\alpha \partial_a x \quad (3.4)$$

(dropping the \mathbb{R}^D indices) where $\lambda^\alpha = \lambda_{AB}^\alpha \in \mathfrak{so}(D)$, and ξ^a denotes any local coordinates on \mathcal{M} . It is useful to supplement them with the "radial" current corresponding to $\lambda^0 = \mathbb{1}$,

$$J_a^0 = x \lambda^0 \partial_a x, \quad J^0 = x_A dx^A = J_a^0 d\xi^a = r dr \quad (3.5)$$

where

$$r^2 = x_A x^A \quad (3.6)$$

is the invariant radius on \mathbb{R}^D . The basic observation underlying this paper is that these currents provide a generalized, over-complete frame for the metric g . More precisely, define the one-forms

$$\theta_a^\alpha = r^{-1} J_a^\alpha, \quad \theta^\alpha = r^{-1} J^\alpha. \quad (3.7)$$

⁴When we write $SO(D)$ usually $SO(1, D-1)$ will also be understood.

Then the following identity⁵ holds

$$g_{ab} = \kappa_{\alpha\beta} \theta_a^\alpha \theta_b^\beta = r^{-2} \kappa_{\alpha\beta} J_a^\alpha J_b^\beta. \quad (3.8)$$

where

$$\kappa^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \text{tr} \lambda^\alpha \lambda^\beta \end{pmatrix} \quad (3.9)$$

is the Killing form of $\mathfrak{so}(D)$ resp. $\mathfrak{so}(1, D-1)$ supplemented by λ^0 . This can be seen using the identity

$$\kappa_{\alpha\beta} \lambda_{AB}^\alpha \lambda_{CD}^\beta = \eta_{AC} \eta_{BD} - \eta_{AD} \eta_{BC} + \eta_{AB} \eta_{CD}, \quad (3.10)$$

which is easy to check for the basis of λ^α given by

$$\lambda_{CD}^{(AB)} = \delta_C^A \delta_D^B - \delta_D^A \delta_C^B, \quad A < B \quad \text{and} \quad \lambda_{CD}^0 = \eta_{CD} \quad (3.11)$$

where $\kappa^{\alpha\beta} = \delta^{\alpha\beta}$ in the Euclidean case. Correspondingly,

$$P^{\alpha\beta} = \theta_a^\alpha \theta_b^\beta g^{ab}, \quad P_\beta^\alpha \theta^\beta = \theta^\alpha \quad (3.12)$$

is a projector on the cotangent bundle $T^*\mathcal{M}$; the frame indices will always be raised and lowered with $\kappa_{\alpha\beta}$, e.g. $P_\beta^\alpha = P^{\alpha\gamma} \kappa_{\beta\gamma}$. The projector on the normal bundle is then given by

$$P_N^{\beta\gamma} = \kappa^{\beta\gamma} - P^{\beta\gamma}, \quad (P_N)^\alpha_\beta \theta^\beta = 0. \quad (3.13)$$

Since the frame θ^α is (over-) complete, any one-form on \mathcal{M} can be written as $v = \sum_\alpha \theta^\alpha v_\alpha = \sum_\alpha \theta^\alpha P_\alpha^\beta v_\beta$. This expansion is unique if we impose that $Pv = v$. In other words, the space of one-forms on \mathcal{M} can be identified with the projective module⁶ $\Omega^1(\mathcal{M}) \cong \mathcal{E}_g := P\mathcal{A}^N$, where $\mathcal{A} = \mathcal{C}(\mathcal{M})$. This construction turns out to be very useful to compute the curvature.

Furthermore, the following identity is shown in appendix B:

$$d\theta^\beta P_\beta^\alpha = -\theta^\beta \omega_\beta^\alpha \quad (3.14)$$

where

$$\omega^{\beta\alpha} = r^{-1}(\theta^\beta P^{0\alpha} - P^{0\beta} \theta^\alpha) = -\omega^{\alpha\beta}. \quad (3.15)$$

which satisfies $P\omega = \omega = \omega P$.

3.3 Connection and curvature

The above realization of the cotangent bundle as a projective module $T^*\mathcal{M} \cong \mathcal{E}_g$ is useful, because it provides a canonical (“Grassmann”) connection

$$\nabla_g = P \circ d : \quad \mathcal{E}_g \rightarrow \mathcal{E}_g \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}). \quad (3.16)$$

⁵No equation of motion or current conservation is needed here.

⁶The label \mathcal{E}_g indicates that the frame θ^α encodes the metric g , to distinguish it from \mathcal{E}_γ introduced below. Module means that the elements can be multiplied (from the right, most naturally) with functions $f \in \mathcal{A}$.

The curvature of this connection is defined by

$$\mathcal{R}[g] = \nabla_g^2 = PdPdP; \quad (3.17)$$

for an introduction to these concepts see e.g. [23]. The last identity follows using $\theta = P\theta$ from

$$\nabla_g^2 \psi = Pd(Pd(P\psi)) = (PdPdP)\psi. \quad (3.18)$$

Under gauge transformations $v \rightarrow \Lambda v \in \mathcal{E}$ which commute with P , the connection transforms as $\nabla_g \rightarrow \nabla'_g = \Lambda \nabla_g \Lambda^{-1} = \nabla_g + P\Lambda^{-1}d\Lambda$, and the curvature $\nabla_g^2 = PdPdP$ transforms in the adjoint. Furthermore, ∇_g is compatible with the inner product on \mathcal{E}_g which arises from $\kappa^{\alpha\beta}$ restricted to $\mathcal{E} = P\mathcal{A}^N$,

$$d(v, w)_g = (\nabla_g v, w)_g + (v, \nabla_g w)_g, \quad (3.19)$$

where

$$(v, w)_g = vw^\dagger = v_\alpha \kappa^{\alpha\beta} w_\beta = v_\alpha P^{\alpha\beta} w_\beta = v_a g^{ab} w_b \quad (3.20)$$

because $Pw = w$, $Pv = v$. This is the usual metric compatibility condition. The gauge transformations are compatible with this inner product if $\Lambda^\dagger = \Lambda^{-1}$, where

$$(v^\dagger)^\alpha = \kappa^{\alpha\beta} v_\beta, \quad (\Lambda^\dagger)_\beta^\alpha = \kappa_{\delta\beta} \Lambda_\gamma^\delta \kappa^{\alpha\gamma} \quad (3.21)$$

etc. Finally, the torsion $T : \mathcal{E} \cong \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A})$ is defined as⁷

$$T(vf) = (d + m \circ \nabla)(vf) = T(v)f \quad (3.22)$$

where $m(\theta \otimes \alpha) = \theta \wedge \alpha$.

Let us compute the Grassmann connection explicitly. The one-form θ^β is represented by

$$\theta^\beta = \theta^\alpha P_\alpha^\beta \cong (P_\alpha^\beta) \in \mathcal{E} = P\mathcal{A}^N \quad (3.23)$$

and its covariant derivative is

$$\nabla_g \theta^\beta = \theta^\alpha \otimes (P_\alpha^\gamma dP_\gamma^\beta). \quad (3.24)$$

Therefore PdP is the connection coefficient. The torsion is given by

$$\begin{aligned} T^\gamma &= d\theta^\gamma + m(\nabla_g \theta^\gamma) = d\theta^\gamma + \theta^\alpha \wedge P_\alpha^\beta dP_\beta^\gamma \\ &= d(\theta^\beta P_\beta^\gamma) + \theta^\beta \wedge dP_\beta^\gamma \\ &= d\theta^\beta P_\beta^\gamma = -\theta^\beta \omega_\beta^\gamma \end{aligned} \quad (3.25)$$

using⁸ the result (3.14). More general connections can be defined as $\nabla \rightarrow \nabla + A$ where $A = A^\alpha_\beta \in \Omega^1(\mathcal{A})$ such that $PA = A = AP$, which is metric compatible if $A^\dagger = -A$. In particular, the torsion vanishes if we choose $A = \omega$,

$$0 = T^\alpha[g] = T^\alpha + \theta^\beta \wedge \omega_\beta^\alpha = 0. \quad (3.26)$$

⁷The present definitions entail $m \circ \nabla(vf) = \nabla(v)f + v \wedge df$ so that gauge covariance holds.

⁸Note that ω is not the spin connection, and the curvature is not given by $d\omega + \omega\omega$.

This is compatible with the metric since $\omega^\dagger = -\omega$ (3.15), and therefore $\nabla[g] = \nabla_g + \omega$ is the Levi-Civita connection on \mathcal{M} for g .

The curvature 2-form \mathcal{R}_β^α is a linear map on $\mathcal{E}_g \cong T^*\mathcal{M}$, which can be written in standard tensorial form using the frame. The Grassmann curvature can be evaluated easily noting that $\theta P_N = 0$ along with $dP = -dP_N$ and $P^\dagger = P$:

$$\begin{aligned}\mathcal{R}[g] &= \nabla[g]^2 = PdPdP \\ \mathcal{R}_{ac}[g] &= \theta_a \mathcal{R} \theta_c^\dagger = \theta_a dP_N dP_N \theta_c^\dagger = d\theta_a P_N d\theta_c^\dagger \\ &= r^{-2} \partial_e J_a^\beta (P_N)_{\beta\gamma} \partial_f J_c^\gamma dx^e dx^f\end{aligned}\tag{3.27}$$

using $d\theta P_N = r^{-1} dJP_N$ in the last step.

Normal embedding coordinates (NEC). To evaluate this, we first choose suitable coordinates at any given point $p \in \mathcal{M}$: using the rotation symmetry of the model, we can assume that $T_p \mathcal{M} \cong (\mathbb{R}^{2n}, 0, \dots, 0)$. We can then choose the first $2n$ matrix components x^a as local coordinates, denoted as “normal embedding coordinates” NEC. It follows that $\partial_{x_A} \partial \partial x^A = 0$ at $p \in \mathcal{M}$, which implies $\partial|_p g_{ab} = 0$. Therefore these are indeed normal coordinates for g in the Riemannian sense, so that we are essentially using $\nabla[g]$. We can furthermore assume $p = (0, \dots, 0, r_0)$ after a suitable translation, so that

$$x_A \partial x^A|_p = r \partial r|_p = J^0|_p = 0.\tag{3.28}$$

Now consider the following tensors

$$\begin{aligned}T_{ab}^\alpha &= \partial_a x \lambda^\alpha \partial_b x \\ K_{ab}^\alpha &= x \lambda^\alpha \nabla_a [g] \partial_b x = x \lambda^\alpha \partial_a \partial_b x|_p = K_{ba}^\alpha,\end{aligned}\tag{3.29}$$

in particular $T_{ab}^0 = g_{ab}$. Clearly K_{ab}^α characterizes the exterior curvature of $\mathcal{M} \subset \mathbb{R}^D$. Then

$$\nabla_a [g] J_b^\alpha = T_{ab}^\alpha + K_{ab}^\alpha = \partial_a J_b^\alpha|_p,\tag{3.30}$$

We note that $T_{ab}^\alpha = -T_{ba}^\alpha$ for $\alpha \neq 0$. It follows using (3.10) that

$$\begin{aligned}K_{ea}^\alpha \kappa_{\alpha\beta} J_d^\beta &= 0 = K_{ea}^\alpha P_\alpha^\beta \\ T_{ea}^\alpha \kappa_{\alpha\beta} J_d^\beta &= J_e^0 g_{ad} - J_a^0 g_{ed} + J_d^0 g_{ea} \\ T_{ea}^\alpha P_\alpha^\beta|_p &= 0 = T_{ea}^\alpha P_{\alpha\beta} T_{fc}^\beta|_p.\end{aligned}\tag{3.31}$$

Therefore⁹ K^α and T^α live in the normal bundle at p , and dropping the contributions of J^0 at p the Grassmann curvature 2-form for the metric g is

$$\boxed{\mathcal{R}[g]_{ac} = r^{-2} \nabla_e J_a \nabla_f J_c^\dagger dx^e dx^f}.\tag{3.32}$$

where $\nabla = \nabla[g]$. The point is that the Grassmann curvature tensor can be expressed in terms of the $\mathfrak{so}(D)$ currents of the matrix model. We now obtain the Riemann tensor for g via

$$\begin{aligned}R[g] &= \nabla[g]^2 = PdPdP + P(d\omega + \omega\omega)P \\ R_{ac}[g] &= \mathcal{R}_{ac}[g] + \theta_a (d\omega + \omega\omega) \theta_c^\dagger \equiv \mathcal{R}_{ac}[g] + \mathcal{R}_{ac}[\omega].\end{aligned}\tag{3.33}$$

⁹Note that $(\nabla_a J_b) J_c^\dagger$ does *not* vanish identically. This is the reason why the final result (3.34) contains additional radial contributions ∇J^0 , and is not obtained trivially by re-shuffling $\nabla J \nabla J$.

Here ω takes care of the radial contributions which are not captured by the $\mathfrak{so}(D)$ currents, as computed in appendix C. Together with the above we obtain

$$\boxed{R[g]_{ac} = r^{-2}(\nabla_e J_a \nabla_f J_c^\dagger - g_{ae} \nabla_f J_c^0 - g_{cf} \nabla_e J_a^0) dx^e dx^f} \quad (3.34)$$

where $\nabla = \nabla[g]$, dropping again contributions of $J^0|_p = 0$ and recalling $\nabla_a J_b^0 = g_{ab} + K_{ab}^0 = \frac{1}{2} \nabla_a \partial_b r^2$. This is the key result, which will be extended to the effective metric γ in the next section. As a check, we proceed by decomposing $\nabla_a J_b^\alpha = T_{ab}^\alpha + K_{ab}^\alpha$ and using

$$r \nabla_a \partial_b r = \frac{1}{2} \nabla_a \partial_b r^2 = \partial_a x_A \partial_b x^A + x_A \nabla_a \partial_b x^A = g_{ab} + x_A \nabla_a \partial_b x^A \quad (3.35)$$

along with the identity (3.10) to obtain

$$\begin{aligned} T_{ea}^\alpha \kappa_{\alpha\beta} T_{fc}^\beta &= (\partial_e x \lambda^\alpha \partial_a x)(\partial_f x \lambda_\alpha \partial_c x) = g_{ef} g_{ac} - g_{ec} g_{af} + g_{ae} g_{cf} \\ K_{ea}^\alpha \kappa_{\alpha\beta} T_{fc}^\beta &= (x \lambda^\alpha \nabla_e \partial_a x)(\partial_f x \lambda_\alpha \partial_c x) = K_{ea}^0 g_{fc} \\ T_{ea}^\alpha \kappa_{\alpha\beta} K_{fc}^\beta &= (\partial_e x \lambda_\alpha \partial_a x)(x \lambda^\alpha \nabla_f \partial_c x) = g_{ea} K_{fc}^0 \\ K_{ea} P_N K_{fc} &= K_{ea}^\alpha \kappa_{\alpha\beta} K_{fc}^\beta = r^2 \nabla_e \partial_a x_A \nabla_f \partial_c x^A. \end{aligned} \quad (3.36)$$

Since we assumed NEC, the coordinate-invariant form is obtained by replacing $\partial_a \rightarrow \nabla_a[g]$. We thus recover the usual Gauss-Codazzi theorem¹⁰ for the Riemann curvature tensor on $\mathcal{M} \subset \mathbb{R}^D$,

$$R[g]_{ac} = d\partial_a x_A d\partial_c x^A = \frac{1}{2}(\nabla_e \partial_a x_A \nabla_f \partial_c x^A - \nabla_f \partial_a x_A \nabla_e \partial_c x^A) dx^e dx^f \quad (3.37)$$

3.4 Effective frame

We now want to develop a similar machinery for the effective metric γ_{ab} on \mathcal{M} . This metric is encoded in the following (over-complete) tangent frame associated to the currents,

$$\begin{aligned} V^\alpha &= x \lambda^\alpha \{x, \cdot\} = J_a^\alpha \theta^{ab} \partial_b \in T\mathcal{M} \\ V^\alpha V^\beta \kappa_{\alpha\beta} &= r^2 \gamma^{ab} \partial_a \otimes \partial_b \end{aligned} \quad (3.38)$$

including $\alpha = 0$ as before. Here $\{., \cdot\}$ is the Poisson bracket on \mathcal{M} , which arises from the non-commutative nature of the brane. However to compute the curvature, it is more natural to use the corresponding frame of one-forms, defined as usual by lowering the index with the effective metric γ . Thus

$$\Theta^\alpha = \Theta_a^\alpha d\xi^a, \quad \Theta_a^\alpha = r^{-1} V^{\alpha,b} \gamma_{ba} = \theta_b^\alpha \mathcal{J}_a^b \quad (3.39)$$

where

$$\mathcal{J}_a^c = \theta^{cb} \gamma_{ba} = \theta_{ab}^{-1} g^{bc}. \quad (3.40)$$

Then the effective metric can be written as

$$\gamma_{ab} = \kappa_{\alpha\beta} \Theta_a^\alpha \Theta_b^\beta = g_{a'b'} \mathcal{J}_a^{a'} \mathcal{J}_b^{b'} = -g_{ac} (\mathcal{J}^2)^c_b \quad (3.41)$$

¹⁰This can be obtained quickly using the projective module defined by the over-complete frame $\theta^A = dx^A$.

and the tangential projector can be expressed in various ways

$$\begin{aligned} P^{\alpha\beta} &= \Theta_a^\alpha \Theta_b^\beta \gamma^{ab} = \theta_c^\alpha \theta_d^\beta \mathcal{J}_a^c \mathcal{J}_b^d \gamma^{ab} = \theta_a^\alpha \theta_b^\beta g^{ab} = -\Theta_a^\alpha \theta^{ae} \theta_e^\beta \\ P^{\alpha\beta} \Theta^\beta &= \Theta^\alpha, \quad P^{\alpha\beta} \theta^\beta = \theta^\alpha. \end{aligned} \quad (3.42)$$

Note that P coincides with the projector defined in the previous section; this is evident due to the relation (3.39) between the frames. The symplectic form Ω on \mathcal{M} is then given by

$$\Theta_a \theta_b^\dagger = \theta_c \mathcal{J}_a^c \theta_b^\dagger = \theta_{ab}^{-1}, \quad \Theta \theta^\dagger = \Omega. \quad (3.43)$$

A cotangent vector can now be written in the two bases as $v = \theta^\alpha v_\alpha = \Theta^\alpha v'_\alpha$ with $Pv = v$, $Pv' = v'$. This gives two different identifications of $T^*\mathcal{M}$ with projective modules \mathcal{E}_g resp. \mathcal{E}_γ . We can determine the transformation $\Lambda v' = v$ between the two frames explicitly, such that

$$\Theta^\alpha = \theta^\beta \Lambda_\beta^\alpha \quad (3.44)$$

and therefore

$$\gamma_{ab} = \Theta_a \Theta_b^\dagger = \theta_a \Lambda \Lambda^\dagger \theta_b^\dagger. \quad (3.45)$$

This Λ is of course not unique. A nice invertible Λ which satisfies this requirement is given by

$$\begin{aligned} \Lambda^{\alpha\beta} &= P_N^{\alpha\beta} + \theta_a^\alpha \theta_b^\beta \Lambda_{(AS)}^{ab} \\ &= \Lambda_{(S)} + \Lambda_{(AS)} \end{aligned} \quad (3.46)$$

where

$$\Lambda_{(AS)}^{ad} = -\mathcal{J}_c^a g^{cd} = g^{ae} g^{dc} \theta_{ec}^{-1} = -\Lambda_{(AS)}^{da} \quad (3.47)$$

$$\Lambda_{(AS)}^\dagger = -\Lambda_{(AS)} \quad (3.48)$$

is anti-symmetric resp. anti-hermitian. It satisfies

$$P\Lambda = P\Lambda_{(AS)} = \Lambda_{(AS)}, \quad P_N\Lambda = P_N. \quad (3.49)$$

The inverse is given explicitly by

$$\Lambda^{-1} = P_N + \theta_a \theta_b \theta^{ab}. \quad (3.50)$$

We will accordingly define $\Lambda_\beta^\alpha = \Lambda^{\alpha\beta'} \kappa_{\beta'\beta}$ etc. Note that the Poisson structure is encoded in Λ , while the embedding is encoded in P . Now consider the Grassmann connection on the projective module \mathcal{E}_γ , given by

$$\begin{aligned} \nabla_\gamma v &= \Theta^\alpha P_\alpha^\beta dv'_\beta = \theta \Lambda P d(\Lambda^{-1} v) \\ \nabla_\gamma &= \Lambda \nabla_g \Lambda^{-1} \end{aligned} \quad (3.51)$$

Therefore ∇_γ is related to ∇_g via the (in general *non-orthogonal*) transformation Λ . This is so because ∇_γ is compatible with the metric γ encoded in $(v, w)_\gamma := v'_\alpha w'_\alpha = v'_\alpha P^{\alpha\beta} w'_\beta$, while ∇_g is compatible with g . The curvature ∇_γ^2 acts on $\mathcal{E}_\gamma \cong T^*\mathcal{M}$ as follows

$$\begin{aligned} \nabla_\gamma^2 v &= \Theta^\beta \mathcal{R}_\beta^\alpha[\gamma] v'_\alpha = \theta \Lambda \mathcal{R}[\gamma] \Lambda^{-1} v \\ \nabla_g^2 v &= \theta \mathcal{R}[g] v \end{aligned} \quad (3.52)$$

reflecting the fact that the connections ∇_γ and ∇_g are related by Λ . As in the previous section, the coordinate form of the (Grassmann) curvature tensor can be obtained using the frame Θ^α

$$\begin{aligned}\mathcal{R}_{ab}[\gamma] &= d\Theta_a P_N d\Theta_b^\dagger = \Theta_a dP_N dP_N \Theta_b^\dagger \\ &= \mathcal{R}_{a'b'}[g] \mathcal{J}_a^{a'} \mathcal{J}_b^{b'} = \theta_a \Lambda dP_N dP_N \Lambda^\dagger \theta_b^\dagger.\end{aligned}\tag{3.53}$$

As explained before, the metric (Levi-Civita) connection corresponding to γ is given by

$$\nabla[\gamma] = \nabla_\gamma + A[\gamma]\tag{3.54}$$

if $A = -A^\dagger$ is such that the torsion vanishes,

$$T[\gamma] = T_\gamma + \Theta A[\gamma] = 0\tag{3.55}$$

To determine A , we compute

$$\begin{aligned}T_\gamma &= d\Theta + m(\nabla_\gamma \Theta) = d\Theta P \\ &= d(\theta \Lambda) P = -\theta d\Lambda P + d\theta P \Lambda \\ &= \Theta(d\Lambda^{-1} \Lambda P - \Lambda^{-1} \omega \Lambda)\end{aligned}\tag{3.56}$$

using $d\theta P = -\theta \omega$ (3.14) and $P\omega = \omega$. Therefore the torsion $T[\gamma]$ vanishes for

$$\begin{aligned}A^\alpha_\beta[\gamma] &= -P d\Lambda^{-1} \Lambda P + P \Lambda^{-1} \omega \Lambda P + \Theta^\alpha B_\beta^{(\alpha)} \\ &= P \Lambda^{-1} d\Lambda P + \Lambda^{-1} \omega \Lambda + \Theta^\alpha B_\beta^{(\alpha)}\end{aligned}\tag{3.57}$$

where $B_\beta^{(\alpha)}$ is arbitrary (since $\Theta^\alpha \kappa_{\alpha\beta} \Theta^\beta = 0$). This is metric compatible if A is anti-hermitian,

$$A^\dagger = -A.\tag{3.58}$$

The second term is always anti-hermitian due to (3.48), (3.15) and $\Lambda^{-1} \omega \Lambda = \Lambda_{(AS)}^{-1} \omega \Lambda_{(AS)}$. In particular, the Grassmann connection is torsion-free if Λ is unitary, which is evident since then the metrics g and γ coincide (3.45).

Conformal rescaling. Now consider the effective metric $G^{ab} = e^{-\sigma} \gamma^{ab}$ (2.10). The above construction can easily be generalized by introducing a suitably rescaled frame

$$\begin{aligned}\tilde{\Theta}_a^\alpha &= e^{-\sigma/2} \Theta_a^\alpha = \theta_b^\alpha \tilde{\mathcal{J}}_a^b = \theta_a^\beta \tilde{\Lambda}_\beta^\alpha \\ \tilde{\mathcal{J}}_a^b &= e^{-\sigma/2} \mathcal{J}_a^b, \quad \tilde{\Lambda} = P_N + e^{-\sigma/2} \Lambda_{(AS)}\end{aligned}\tag{3.59}$$

such that

$$\tilde{\Theta}_a^\alpha \tilde{\Theta}_b^\beta \kappa_{\alpha\beta} = G_{ab}.\tag{3.60}$$

This leaves the projector P unchanged. However this kind of rescaling is more appropriate after compactification, and we will largely work with γ^{ab} in this paper.

3.5 Special geometry

In general, we cannot give an explicit form for the $B_\beta^{(\alpha)}$ required for the Levi-Civita connection. We therefore restrict ourselves to a certain class of preferred geometries. More specifically, we consider geometries with

$$\nabla[g]Q \equiv PdQP = 0 \quad (3.61)$$

where

$$\begin{aligned} Q &:= \Lambda\Lambda^\dagger - \mathbb{1} = -(\Lambda_{(AS)}^2 + P) = -\theta_a(\mathcal{J}^2 + \delta)^a_b g^{bc} \theta_c^\dagger = \Lambda^\dagger \Lambda - \mathbb{1} \\ &= PQ = QP. \end{aligned} \quad (3.62)$$

Q measures the deviation from \mathcal{J} being an almost-complex structure, in particular $Q = 0$ for almost-Kähler geometries (in the Euclidean case). Together with (3.41) and $\nabla Q = \theta \nabla \mathcal{J}^2 g \theta^\dagger$ this implies $\nabla[g]\gamma = \nabla[g](g\mathcal{J}^2) = 0$, so that this condition is equivalent to

$$\nabla \mathcal{J}^2 = 0, \quad \nabla[g] \equiv \nabla[\gamma] \equiv \nabla[G] \equiv \nabla. \quad (3.63)$$

The last equality follows from $\partial \det \mathcal{J}^2 = 0$ together with (2.10). This means that the connections on \mathcal{M} defined by γ and g and G are equivalent, which is very reasonable. Now $PdQP = 0$ implies

$$\begin{aligned} 0 &= Pd\Lambda\Lambda^\dagger P + P\Lambda d\Lambda^\dagger P \\ 0 &= P\Lambda^{-1}d\Lambda P + P(\Lambda^{-1}d\Lambda)^\dagger P \end{aligned} \quad (3.64)$$

so that

$$A_\Lambda := P\Lambda^{-1}d\Lambda P = \Lambda_{(AS)}^{-1}d\Lambda_{(AS)} = -A_\Lambda^\dagger, \quad (3.65)$$

using $0 = PdPP \equiv \nabla_g P$, and the Levi-Civita connection $\nabla[\gamma]$ is obtained for $B_\beta^{(\alpha)} = 0$. Note that we do *not* require $\mathcal{J}^2 = -1$, which is impossible in the Minkowski case due to the inequivalent causal structures of g and γ . However $\nabla \mathcal{J}^2 = 0$ is compatible with a Minkowski signature, and milder¹¹ than $\nabla \mathcal{J} = 0$. Typically \mathcal{J}^2 defines an (integrable) decomposition of $T\mathcal{M}$ into rank 2 sub-bundles. Moreover, it is not hard to see that the equations of motion for the Poisson structure θ^{ab} derived from the bosonic action

$$S_{\text{YM}} \sim \int d^{2n} \xi \sqrt{|\theta^{-1}|} \gamma^{ab} g_{ab} = - \int \Omega^{\wedge n} \text{tr} \mathcal{J}^{-2} \quad (3.66)$$

are always satisfied if $\nabla \mathcal{J}^2 = 0$; this will become clear in section 3.7. Moreover, geometries with $\nabla \mathcal{J}^2 = 0$ are not only solutions but are expected to be preferred “ground state” solutions for the Poisson structure. This is true at least for 4-dimensional Euclidean branes, where the bosonic action is positive definite and takes its minimum if and only $\mathcal{J}^2 = -\delta$ i.e. $Q = 0$ [8].

We therefore expect that $\nabla \mathcal{J}^2 = 0$ will always hold at least asymptotically. However in general, $\nabla \mathcal{J}^2 = 0$ might not always be compatible with a given g , and matter might lead to short-range perturbations of \mathcal{J}^2 or θ^{ab} . As observed by Rivelles [17], such perturbations are in fact Ricci-flat at least on \mathbb{R}^4 . Thus we expect that the Poisson structure is adjusted dynamically such that $\nabla \mathcal{J}^2 \approx 0$, and possible deviations from $\nabla \mathcal{J}^2 = 0$ are suppressed for long distances and could be treated perturbatively. Special geometry should be even less restrictive in the presence of compactified extra dimensions, an compatible with all physically relevant 4-dimensional effective geometries. The dynamics of \mathcal{J} will be studied in section 3.7.

¹¹which in turn is milder than e.g. the Kähler condition since $\mathcal{J}^2 \neq -1$.

3.6 Curvature and effective gravity

Let us therefore assume special geometries with $\nabla Q = 0$. Then the Levi-Civita connection ∇ is given by

$$\nabla = \nabla_\gamma + A, \quad A = \Lambda_{(AS)}^{-1} d\Lambda_{(AS)} + \omega \quad (3.67)$$

using (3.46). After some algebra, we obtain the following expression for the Riemann curvature for γ (see appendix D) using (3.53),

$$\begin{aligned} R_{ab}[\gamma] &= \theta_a(dP_N dP_N + d\omega + \omega\omega)\Lambda\Theta_b^\dagger \\ &= -R_{ab'}[g]\mathcal{J}^{2b'}_b = -(\mathcal{R}_{ab'}[g] + \mathcal{R}_{ab'}[\omega])\mathcal{J}^{2b'}_b \end{aligned} \quad (3.68)$$

recalling that $\Lambda_{(AS)}^\dagger = -\Lambda_{(AS)}$. This also follows from¹² $R^a_b[\gamma] \equiv R^a_b[g]$, since $\nabla[\gamma] = \nabla[g]$ for special geometries. On the other hand, it follows from (3.67) that $dA_\Lambda + A_\Lambda A_\Lambda = 0$, so that the Riemannian curvature for γ can be obtained from the Grassmann curvature via

$$\begin{aligned} R_{ab}[\gamma] &= \mathcal{R}_{ab}[\gamma] + \Theta_a \Lambda^{-1}(d\omega + \omega\omega)\Lambda\Theta_b^\dagger \\ &= \mathcal{R}_{a'b'}[g]\mathcal{J}^{a'}_a \mathcal{J}^{b'}_b - \theta_a(d\omega + \omega\omega)\theta_b^\dagger \mathcal{J}^{2b'}_b \end{aligned} \quad (3.69)$$

using (3.53) in the second line. These are explicit and compact expressions for the effective curvature, which together with the expression (3.32) for $\mathcal{R}_{ab}[g]$ in terms of the currents constitutes a main result of this paper. Comparing the two results (3.69) and (3.68) implies $\mathcal{R}_{a'b'}[g]\mathcal{J}^{a'}_a \mathcal{J}^{-1b'}_b = -\mathcal{R}_{ab}[g]$, and noting that $\mathcal{R}[\omega]$ satisfies the standard symmetries of the Riemann tensor (e.g. using the explicit form (C.3)) we have

$$\mathcal{R}_{a'b';cd}[g]\mathcal{J}^{a'}_a \mathcal{J}^{-1b'}_b = -\mathcal{R}_{ab;cd}[g] = \mathcal{R}_{ab;c'd'}[g]\mathcal{J}^{c'}_c \mathcal{J}^{-1d'}_d. \quad (3.70)$$

Now we compute the Ricci tensor from (3.68):

$$\text{Ric}_{ac}[\gamma] = \gamma^{bd} R_{ab;cd}[\gamma] = g^{bd} \mathcal{R}_{ab;cd}[g] + g^{bd} \mathcal{R}_{ab;cd}[\omega] = \text{Ric}_{ac}[g]. \quad (3.71)$$

Consider the two terms separately. For the first term, we use the relation (3.70) as follows

$$g^{bd} \mathcal{R}_{ab;cd}[g] = g^{bd} \mathcal{R}_{a'b';c'd'}[g]\mathcal{J}^{a'}_a \mathcal{J}^{-1b'}_b \mathcal{J}^{c'}_c \mathcal{J}^{-1d'}_d = \gamma^{bd} \mathcal{R}_{a'b';c'd'}[g]\mathcal{J}^{a'}_a \mathcal{J}^{c'}_c. \quad (3.72)$$

Now we can use the explicit form of $\mathcal{R}[g]$ in terms of the currents is given by (3.32), and together with (3.36) and (3.35) we obtain in NEC

$$\begin{aligned} \nabla_a J_b^0 &= g_{ab} + K_{ab}^0 = r \nabla_a \partial_b r \\ \nabla^d J_d T_{ac}^\dagger &= \gamma^{bd}(T + K)_{bd} T_{ac}^\dagger = \gamma^{bd} r \nabla_b \partial_d r g_{ac} = (\gamma^{bd} \nabla_b J_d^0) g_{ac} \\ \gamma^{bd} \nabla_d J_b \nabla_c J_a^\dagger &= \partial_d (\gamma^{db} J_b) \nabla_c J_a^\dagger = \nabla^d J_d K_{ac}^\dagger + (\gamma^{bd} \nabla_b J_d^0) g_{ac} \\ &= \gamma^{bd} K_{db} K_{ac}^\dagger + \gamma^{bd} K_{db}^0 g_{ac} + (\gamma^{bd} g_{bd}) \nabla_a J_c^0 \\ \gamma^{bd} \nabla_c J_b \nabla_d J_a^\dagger &= \gamma^{bd} (T_{bc} + K_{bc})(T_{da}^\dagger + K_{da}^\dagger). \end{aligned} \quad (3.73)$$

¹²Note that the non-trivial perturbations of γ on \mathbb{R}_θ^4 due to fluctuations of the Poisson structure discussed e.g. in [6, 17] are not compatible with the assumption of special geometry, so there is no contradiction.

using $J^0|_p = 0$. To proceed, we assume a compactified brane of the form $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$ where \mathcal{K} has a *small* radius of scale r_K , much smaller than any scale $r_{\mathcal{M}}$ associated with the non-compact part. Then the dominant terms are those arising from the extrinsic curvature on \mathcal{K} , which is $K_{ab}K_{cd} \sim r_K^{-2}$. Therefore we only keep the terms quadratic in K_{ab}^α from now on and drop the rest, so that

$$\begin{aligned}\gamma^{bd}\nabla_d J_b \nabla_c J_a^\dagger &= \nabla^d J_d K_{ac}^\dagger + \mathcal{O}\left(\frac{r_K}{r_{\mathcal{M}}}\right) \\ &= -\Lambda_0^{-4} T_{cd} \Pi_{ef}^{cd} \theta^{ee'} \theta^{ff'} K_{e'f'} K_{ac}^\dagger + \mathcal{O}\left(\frac{r_K}{r_{\mathcal{M}}}\right) \\ \gamma^{bd}\nabla_c J_b \nabla_d J_a^\dagger &= \gamma^{bd} K_{bc} K_{da}^\dagger + \mathcal{O}\left(\frac{r_K}{r_{\mathcal{M}}}\right)\end{aligned}\tag{3.74}$$

using current conservation (4.11). Furthermore, the contributions (C.4) from $g^{ae} \mathcal{R}_{ac;ef}[\omega]$ are negligible in the same approximation. Then the Ricci tensor for the effective metric is obtained from (3.71) as

$$\begin{aligned}\text{Ric}_{ac}[\gamma] &= r^{-2} \left(-\Lambda_0^{-4} T_{cd} \Pi_{ef}^{cd} \theta^{ee'} \theta^{ff'} K_{e'f'} K_{a'd}^\dagger - \gamma^{bd} K_{bc'} K_{a'd}^\dagger \right) \mathcal{J}_a^{a'} \mathcal{J}_c^{c'} \\ &= r^{-2} \left(\Lambda_0^{-4} T_{cd} \Pi_{ef}^{cd} \theta^{ee'} \theta^{ff'} K_{e'f'} K_{a'd}^\dagger + \gamma^{bd} K_{bc'} K_{a'd}^\dagger \right) \mathcal{J}_c^{2c'}\end{aligned}\tag{3.75}$$

where the second form follows directly from (3.68). The first line becomes more appealing (and more appropriate for the reduction to 4 dimensions, as explained below) in upper-component notation. Using also $\partial e^\sigma = 0$, we obtain a compact expression for the Ricci tensor provided $\nabla \mathcal{J}^2 = 0$,

$$\boxed{e^{2\sigma} \text{Ric}^{ac}[G] = \text{Ric}^{ac}[\gamma] = -T_{b'd'} \Pi_{bd}^{b'd'} \mathcal{P}^{bd;ac} - \Lambda_0^4 g_{bd} \mathcal{P}^{ab;cd} + \mathcal{O}\left(\frac{r_K}{r_{\mathcal{M}}}\right)},\tag{3.76}$$

refining¹³ the previous results in [20]. However to fully understand the effective gravity on \mathcal{M} we need to understand also the response of the second term $g_{bd} \mathcal{P}^{ab;cd}$ to matter, which might contain an additional hidden coupling to T_{ab} . Therefore this equation does not allow to draw immediate physical conclusions. Nevertheless, the message is that the Ricci tensor is related to the energy-momentum tensor of matter, *without* invoking an Einstein-Hilbert-type action or quantum effects. The coupling of matter to the Ricci-tensor is mediated by the tensor

$$\begin{aligned}\mathcal{P}^{cd;ab} &= r^{-2} \Lambda_0^{-4} \theta^{cc'} \theta^{dd'} K_{c'd'} K_{a'b'}^\dagger \theta^{aa'} \theta^{bb'} = \Lambda_0^{-4} \theta^{cc'} \theta^{dd'} \theta^{aa'} \theta^{bb'} \partial_{c'} \partial_{d'} x^A \partial_{a'} \partial_{b'} x_A \\ \Pi_{ab}^{cd} &= \delta_{ab}^{cd} - \frac{1}{2(n-1)} \gamma_{ab} \gamma^{cd}\end{aligned}\tag{3.77}$$

which is determined by the extrinsic curvature of the embedding $\mathcal{M} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$ and the Poisson tensor θ^{ab} . The second form of \mathcal{P} follows from (3.36). Without extrinsic curvature, \mathcal{P} would vanish, and not even Newtonian gravity would arise¹⁴. The last terms subsume the “mixing terms” which remained mysterious in [20].

The expression (3.76) should be a suitable starting point to study the effective gravity on branes, which will be pursued elsewhere. However we emphasize several points here. First,

¹³The sign appears to be inconsistent with [20].

¹⁴Of course other mechanisms are conceivable such as induced gravity or holography. However, then the usual fine-tuning problems would arise.

the extrinsic curvature is necessarily large on the compact extra dimensions \mathcal{K} , which is transmitted to the *non-compact* space \mathcal{M}^4 by the Poisson tensor as in (2.15). In this way, the compactification moduli of \mathcal{K} can play the role of gravitational degrees of freedom for \mathcal{M}^4 , and their origin in the spontaneously broken global symmetries of the matrix model implies that they remain massless¹⁵. Such Poisson tensors $\theta^{\mu i}$ which relate the compact with the non-compact space naturally arise on compactified brane solutions in matrix models, dubbed split non-commutativity [22]. For example, a spherical compactification $\mathcal{K} = S^2 \subset \mathbb{R}^6$ would lead to $K_{ij}K_{kl}^\dagger = r_{\mathcal{K}}^{-2}\delta_{ij}\delta_{kl}$, which is too simple to provide full Einstein gravity. However the \mathcal{K} typically has to rotate along \mathcal{M}^4 in order to be a solution [22], and there are plenty of more sophisticated compactifications [25]. Moreover, we only need (near-) Einstein gravity in the 4 non-compact direction, and not on \mathcal{K} . It remains to be seen if a realistic 4-dimensional gravity can be obtained for suitable compactifications. If so, this could provide a very appealing theory for gravity which is not only well-suited for quantization, but also protected from the usual fine-tuning problems.

It should be clear that this mechanism is completely unavoidable on branes of the structure $\mathcal{M} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$ in matrix models. It implies a long-range gravity-like force on \mathcal{M}^4 , which would certainly dominate the bulk gravity with its r^{-8} Newton law at long distances. Hence there is no need for 10-dimensional compactification, and the selection of the present type of compactification is a well-defined and predictive question within the matrix model.

Finally, we recall that short-range perturbations with $\nabla \mathcal{J}^2 \neq 0$ are expected in the presence of matter, as discussed in section 3.7.

Towards 4-dimension gravity. Although the above results apply to any $\mathcal{M} \subset \mathbb{R}^D$, we are mainly interested in the low-energy sector on backgrounds of the form $\mathcal{M} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$. We should therefore perform an appropriate reduction on \mathcal{K} , and study the 4-dimensional effective geometry. This reduction is not trivial here, because \mathcal{K} is *not* perpendicular to \mathcal{M}^4 . As discussed in [20], the effective 4-dimensional metric with upper (!) indices $G^{\mu\nu}$ is obtained from G^{ab} by dropping the extra coordinates¹⁶, and averaging over \mathcal{K} if necessary

$$G_{4D}^{\mu\nu} := \int_{\mathcal{K}} G^{\mu\nu} . \quad (3.78)$$

Here we assume that the low-energy physical fields are constant along \mathcal{K} (for the lowest KK modes), which moreover has constant radius as discussed in section 3.8. However, the inverse effective 4-dimensional metric $G_{\mu\nu}^{4D}$ is in general *not* such a simple reduction of G_{ab} ; non-compact coordinates are indicated by Greek letters. We should thus be careful before drawing physical conclusions, but the salient features are expected to survive. In particular, the term

$$\gamma^{bd} K_{ac} K_{bd}^\dagger \quad (3.79)$$

is certainly large on \mathcal{K} but should typically vanish on \mathcal{M}^4 , consistent with the fact that the 4-dimensional geometry is flat for the basic solutions found in [22, 25], in the absence of matter. In particular, assuming that $\mathcal{P}^{ab;cd}$ is Lorentz-invariant with respect to the 4-dimensional

¹⁵This is not the case for the radial modes, which were discussed in [20]. These are in fact assumed to be massive here due to the flux on \mathcal{K} .

¹⁶Due to the flux stabilization mechanism, we can assume here that $r_{\mathcal{K}} = \text{const}$ as discussed below.

effective metric¹⁷ and assuming that the properly reduced equations (3.76) have the same form, we would indeed obtain the Einstein equations, possibly with additional vacuum contributions due to $g_{\eta\sigma}\mathcal{P}^{\mu\eta;\nu\sigma}$. The effective gravitational constant is set by the scales of compactification and Λ_0 [20],

$$G_N \sim r_{\mathcal{K}}^{-2}\Lambda_0^{-4} . \quad (3.80)$$

Although this requires several assumptions about the background (most importantly effective Lorentz invariance of \mathcal{P}), the message is that an effective gravity similar to Einstein gravity can arise from compactified branes $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$ in matrix models, without an Einstein-Hilbert action. The physical meaning of the additional term $g_{\eta\sigma}\mathcal{P}^{\mu\eta;\nu\sigma}$ remains to be clarified. It might contribute an additional coupling to $T_{\mu\nu}$, and it will probably contribute constant tensors such as $\gamma^{\mu\nu}$ or $(\gamma g \gamma)^{\mu\nu}$. Furthermore, harmonic contributions from \mathcal{M}^4 may also play a role, cf. [19]. It is tempting to speculate that these modifications of the Einstein equations might manifest themselves as dark matter and/or energy.

3.7 Energy-momentum conservation

To understand better possible deviations from $\nabla \mathcal{J}^2 = 0$, we study the tangential degrees of freedom in more detail. These are conveniently captured by the matrix conservation law [6]

$$0 = -i[X_B, \mathcal{T}^{AB}] \sim \{x_B, \mathcal{T}^{AB}\} \quad (3.81)$$

where \mathcal{T}^{AB} is the "matrix" energy-momentum tensor. Dropping the contributions of the spinorial (fermionic) matrices Ψ , it is given explicitly by

$$\mathcal{T}^{AB} = \frac{1}{2}([X^A, X^C][X^B, X_C] + (A \leftrightarrow B)) - \frac{1}{4}\eta^{AB}[X^C, X^D][X_C, X_D]. \quad (3.82)$$

We can split this tensor into geometrical and matter content,

$$\begin{aligned} \mathcal{T}^{AB} &= \mathcal{T}_{\text{geom}}^{AB} + \mathcal{T}_{\text{mat}}^{AB} = \partial_a x^A \partial_b x^B \theta^{aa'} \theta^{bb'} \left(T_{a'b'}^{\text{geom}} + e^\sigma \Lambda_0^{-4} T_{a'b'}^{\text{mat}} \right), \\ T_{ab}^{\text{geom}} &= -g_{ab} + \frac{1}{4} \gamma_{ab} (\gamma^{cd} g_{cd}) \end{aligned} \quad (3.83)$$

and the nonabelian component is essentially the usual energy-momentum tensor, at least for $\theta = \text{const.}$ Therefore

$$\{x_B, \mathcal{T}_{\text{geom}}^{AB}\} = -\{x_B, \mathcal{T}_{\text{mat}}^{AB}\} \quad (3.84)$$

describes the back-reaction of matter to the Poisson structure. To understand this, we observe

$$\begin{aligned} \{x_B, \mathcal{T}^{AB}\} &= \theta^{cd} \partial_c x^B \partial_d (\mathcal{T}^{AD}) \eta_{BD} \\ &= \theta^{cd} \partial_d (\partial_a x^A \partial_c x^B \partial_b x^D \theta^{aa'} \theta^{bb'} T_{a'b'}) \eta_{BD} \\ &= \theta^{cd} \partial_d (g_{cb} \theta^{bb'} T_{a'b'} \theta^{aa'} \partial_a x^A) \end{aligned} \quad (3.85)$$

¹⁷Lorentz invariance with respect to the full metric on \mathcal{M} is presumably too restrictive, and we do not expect that Einstein gravity is recovered on \mathcal{M}^{2n} . Moreover only a part of the full tensor \mathcal{P} is used after the reduction, so that even the effective sign is not clear at this point.

in any local coordinates. For the geometric contribution, this can be written as

$$\begin{aligned}\{x_B, \mathcal{T}_{\text{geom}}^{AB}\} &= \theta^{cd} \partial_d (\mathcal{J}^{-2} - \frac{1}{4} (\text{tr} \mathcal{J}^{-2}) \delta)_c^a \partial_a x^A \\ &= \theta^{cd} (\nabla_d \mathcal{J}_c^{-2a} - \frac{1}{4} \partial_d (\text{tr} \mathcal{J}^{-2}) \delta_c^a) \partial_a x^A\end{aligned}\quad (3.86)$$

(for any torsion-free ∇) noting that the transversal contribution vanish, in particular

$$\gamma^{db} g_{a'b} \theta^{aa'} \nabla_d \partial_a x^A = (\theta^{dd'} g_{d'b'} \theta^{bb'} g_{ba} \theta^{ac}) \nabla_d \partial_c x^A = 0 \quad (3.87)$$

since $\theta g \theta g \theta$ is anti-symmetric. Thus (3.86) is purely tangential. For the matter contribution, we can proceed as follows

$$\begin{aligned}\Lambda_0^4 \{x_B, \mathcal{T}_{\text{matter}}^{AB}\} &= \frac{e^\sigma}{\sqrt{G}} \partial_d (\sqrt{G} G^{db} T_{ab} \theta^{a'a} \partial_{a'} x^A) \\ &= e^\sigma \left(G^{db}(x) \nabla_d [G] T_{ba} - \frac{1}{2} \partial_a G^{db} T_{bd} \right) \theta^{a'a} \partial_{a'} x^A + e^\sigma G^{db} T_{ab} \partial_d (\theta^{a'a} \partial_{a'} x^A) \\ &\stackrel{\text{cons}}{=} e^\sigma \left(-\frac{1}{2} T_{bd} \nabla_a [g] G^{db} \mathcal{J}_e^{-1a} + G^{db} T_{ba} \nabla_d [g] \mathcal{J}_e^{-1a} \right) g^{ea'} \partial_{a'} x^A + (\dots) \nabla [g] \partial x^A\end{aligned}$$

using the identity [6]

$$\rho \nabla_b \theta^{bc} = \theta^{cb} \partial_b \rho, \quad \rho = \sqrt{|\theta^{-1}|} \quad (3.88)$$

and (E.4) in the appendix. The first two lines hold in any coordinates, and energy-momentum conservation $\nabla^b [G] T_{ba} = 0$ was assumed in the last line. We choose normal embedding coordinates such that $\partial \partial x^A = \nabla [g] \partial x^A$ is in the normal bundle, and together with (3.86) the tangential components give

$$\theta^{cd} \nabla_d \mathcal{J}_c^{-2a} - \frac{1}{4} \theta^{ad} \partial_d (\text{tr} \mathcal{J}^{-2}) = -\frac{e^\sigma}{2} \Lambda_0^{-4} T_{bd} \nabla_c [g] G^{db} \mathcal{J}_e^{-1c} g^{ae} + e^\sigma \Lambda_0^{-4} G^{db} T_{bc} \nabla_d [g] \mathcal{J}_e^{-1c} g^{ae}. \quad (3.89)$$

Therefore any vacuum geometry with $\nabla \mathcal{J}^2 = 0$ is a solution. Short-range perturbations of $\nabla \mathcal{J}^2$ are expected in the presence of matter, which do not significantly contribute to gravity at long distances. To see this, it is better to use the fundamental degrees of freedom given by the Poisson structure and the embedding. Using the identity (E.7), the same conservation law can be written as follows

$$\gamma^{da} \nabla_a [g] \theta_{bd}^{-1} = \frac{e^\sigma}{2} \Lambda_0^{-4} T_{bd} \nabla_c [g] G^{db} \theta^{ca} \gamma_{ba} - e^\sigma \Lambda_0^{-4} G^{db} T_{bc} \nabla_d [g] \theta^{ca} \gamma_{ba}. \quad (3.90)$$

Since this has the structure of Maxwell equations, the perturbations of θ_{bd}^{-1} due to matter decay at least as $\frac{1}{r^2}$, and therefore do not contribute to gravity at long distances. This is consistent¹⁸ with the equation (3.6) in [20], which was obtained directly from the action.

¹⁸The assumption $\Gamma^a = 0$ in [20] amounts to $\{x_B, \mathcal{T}_{\text{geom}}^{AB}\} = 0$ via (E.5), and therefore follows from $\nabla \mathcal{J}^2 = 0$.

3.8 Radial equation of motion and flux stabilization

The equation of motion for the radial mode $r^2(x) = x^A x_A$ can be derived using the identity (3.35), which gives

$$\frac{1}{2}\square r^2 = r\square r = \gamma^{ab}(g_{ab} + K_{ab}^0) = \gamma^{ab}g_{ab} - \Lambda_0^{-4}T_{cd}\Pi_{a'b'}^{cd}\theta^{a'a}\theta^{b'b}K_{ab}^0 \quad (3.91)$$

Since we argued or assumed that $\nabla\mathcal{J}^2 = 0$ to a very good approximation, it follows that $(\gamma g) = -\text{tr}\mathcal{J}^{-2} = \text{const.}$ This vanishes if and only if the action is invariant under $x^A \rightarrow \alpha x^A$, and one may expect that this is preferred upon quantization.

Now consider the case of compactified extra dimensions $M^4 \times \mathcal{K} \subset \mathbb{R}^{10}$ where $\mathcal{K} \subset \mathbb{R}^6$ is compact. We can locally write $\mathbb{R}^{10} = \mathbb{R}^4 \times \mathbb{R}^6$ with $x^A = (x^\mu, y^i)$ such that the radius is $r_{\mathcal{K}}^2 = y_i y^i$, and use the 4 non-compact matrices x^μ as part of the local coordinates $\xi^a = (x^\mu, \xi^i)$. Then the equation of motion for $r_{\mathcal{K}}$ can be obtained as follows:

$$\square r^2 = (\gamma^{\mu\nu}\partial_\mu\partial_\nu)(x^\rho x^\sigma \eta_{\rho\sigma}) + \square r_{\mathcal{K}}^2 = 2\gamma^{\mu\nu}\eta_{\mu\nu} + \square r_{\mathcal{K}}^2 \quad (3.92)$$

in NEC. Together with the above we obtain

$$\begin{aligned} \frac{1}{2}\square r_{\mathcal{K}}^2 &= 2\gamma^{i\mu}g_{i\mu} + \gamma^{ij}g_{ij} \\ &= 2\gamma^{i\mu}g_{i\mu} + g_{ij}\theta^{ii'}\theta^{jj'}g_{i'j'} = f(r_{\mathcal{K}}) \end{aligned} \quad (3.93)$$

in vacuum. This is a polynomial in $r_{\mathcal{K}}$ via $g_{ij} \sim r_{\mathcal{K}}^2$. If the flux θ^{ij} on \mathcal{K} does not vanish, then the rhs contains quadratic and quartic terms in $r_{\mathcal{K}}$, and will vanish for a certain radius r_0 for suitable $\theta^{\mu i}$. The radial perturbations of the compactification \mathcal{K} are then in general stabilized by the flux and (very) massive, so that we can safely set $r_{\mathcal{K}} = \text{const}$ at low energies. This is the flux stabilization mechanism in the present context.

4 Perturbations of the geometry

Consider a perturbation

$$x^A \rightarrow x^A + \delta x^A \quad (4.1)$$

of some background brane $\mathcal{M}^{2n} \subset \mathbb{R}^D$, defined in terms of matrices $X^A \sim x^A$ as above. We can certainly describe the most general such deformations in the form

$$\delta x^A = -\sum_{\alpha \neq 0} \epsilon_\alpha (\lambda^\alpha x)^A + \delta r x^A \quad (4.2)$$

where $\epsilon_\alpha = \epsilon_\alpha(x)$ and $\epsilon_0 \equiv \delta r = \delta r(x)$ are arbitrary functions. This is of course an over-parametrization. The corresponding metric perturbation can be written in terms of the currents as

$$\begin{aligned} \delta g_{ab} &= -\partial_a x \partial_b (\epsilon_\alpha(x) \lambda^\alpha x) + \partial_a x \partial_b (\delta r(x) x) + (a \leftrightarrow b) \\ &= J_a^\alpha \partial_b \epsilon_\alpha + J_b^\alpha \partial_a \epsilon_\alpha + 2\epsilon_0 g_{ab} \end{aligned} \quad (4.3)$$

since $\lambda^{\alpha \neq 0}$ is anti-symmetric. To clarify the relation with the approach in [20], we can then rewrite this as

$$\delta g_{ab} = \nabla_b(\epsilon^\alpha J_a^\alpha) - \epsilon^\alpha \nabla_b J_a^\alpha + \epsilon_0 g_{ab} + (a \leftrightarrow b) \quad (4.4)$$

$$= -2\epsilon_\alpha K_{ab}^\alpha + \nabla_a V_b^\epsilon + \nabla_b V_a^\epsilon + 2\epsilon_0 g_{ab} \quad (4.5)$$

where $\nabla = \nabla[g]$. Then the vector fields

$$V_b^\epsilon = \epsilon^\alpha J_b^\alpha \quad (4.6)$$

encode the tangential perturbations, while the extrinsic curvature of $\mathcal{M} \subset \mathbb{R}^D$ leads to linearized metric perturbations $-2\epsilon_\alpha K_{ab}^\alpha$ due to transversal brane perturbations.

4.1 Current conservation and matter

In the presence of matter, the $SO(D)$ rotations also act on the fermions and gauge fields. Rather than trying to derive the generalized currents, we want to incorporate matter as source term for the conservation law of the geometrical current (3.2). We therefore need the variation of the matter action under the local perturbations (4.2) acting only on the geometry defined by the $U(1)$ sector of the matrices $X^A \sim x^A$, in the presence of fixed matter fields resp. matrices (on-shell). Restricting ourselves to the semi-classical case, matter couples to the background as usual via the effective metric G . Therefore the variation of the action under these geometrical $SO(D)$ rotations is simply obtained by the energy-momentum tensor T_{ab} of matter coupled to δG_{ab} . We choose to work in Darboux coordinates where $\theta^{ab} = \text{const}$ is fixed, which is always possible¹⁹. Then the variation of the effective metric (2.10) takes the form

$$\delta G^{ab} = e^{-\sigma} \Pi_{cd}^{ab} \theta^{cc'} \theta^{dd'} \delta g_{c'd'} \quad (4.7)$$

where

$$\Pi_{ab}^{cd} = \delta_{ab}^{cd} - \frac{\gamma_{ab} \gamma^{cd}}{2(n-1)} \quad (4.8)$$

Then

$$\begin{aligned} \delta S_{\text{YM}} + \delta S_{\text{matter}} &= \frac{1}{2(2\pi)^n} \int d^{2n}x \left(\Lambda_0^4 \sqrt{\theta^{-1}} \gamma^{ab} \delta g_{ab} + \sqrt{G} T_{ab} \delta G^{ab} \right) \\ &= \frac{1}{2(2\pi)^n} \int d^{2n}x \sqrt{\theta^{-1}} \left(\Lambda_0^4 \gamma^{ab} + T_{cd} \Pi_{a'b'}^{cd} \theta^{a'a} \theta^{b'b} \right) \delta g_{ab} \end{aligned} \quad (4.9)$$

where δg_{ab} is given by (4.3). Upon partial integration, we obtain the current conservation law in the presence of matter

$$\begin{aligned} \partial_a(\gamma^{ab} J_b^\alpha) &= -\Lambda_0^{-4} \partial_a(T_{cd} \Pi_{a'b'}^{cd} \theta^{a'a} \theta^{b'b} J_b^\alpha), \quad \alpha \neq 0 \\ \partial_a(\gamma^{ab} J_b^0) &= -\Lambda_0^{-4} T_{cd} \Pi_{a'b'}^{cd} \theta^{a'a} \theta^{b'b} K_{ab}^0 + (\gamma^{ab} g_{ab}) \end{aligned} \quad (4.10)$$

¹⁹From the point of view of noncommutative gauge theory on \mathbb{R}_θ^{2n} , this means that all matter fields and $SU(n)$ -valued fields are fixed, and only the trace- $U(1)$ scalar fields are perturbed. The latter are interpreted as perturbations of the embedding metric δg_{ab} .

The second equation follows recalling that $\partial_a J_b^0 = g_{ab} + K_{ab}^0$ (3.30), and setting $J^0|_P \sim \partial r|_P = 0$ after a suitable translation. The lhs can be written covariantly using (A.8), and we obtain

$$\begin{aligned} e^\sigma \nabla^a [G] J_a^\alpha &= \gamma^{ab} K_{ab}^\alpha + \mathcal{O}(J^\alpha) = -\Lambda_0^{-4} T_{cd} \Pi_{a'b'}^{cd} \theta^{a'a} \theta^{b'b} K_{ab}^\alpha + \mathcal{O}(J^\alpha), \quad \alpha \neq 0 \\ e^\sigma \nabla^a [G] J_a^0 - (\gamma^{ab} g_{ab}) &= \gamma^{ab} K_{ab}^0|_P = -\Lambda_0^{-4} T_{cd} \Pi_{a'b'}^{cd} \theta^{a'a} \theta^{b'b} K_{ab}^0 \end{aligned} \quad (4.11)$$

Note that $\mathcal{O}(J^\alpha)$, $\alpha \neq 0$ drops out from the equation (3.76) for the Ricci tensor because it is tangential, while K_{cd}^α is transversal. The basic mechanism can now be seen by observing that current conservation (3.3)

$$\begin{aligned} e^\sigma \nabla^a [G] J_a^\alpha &= \gamma^{ab} K_{ab}^\alpha = x \lambda^\alpha \square_G x, \quad \alpha \neq 0, \\ e^\sigma \nabla^a [G] J_a^0 - \gamma^{ab} g_{ab} &= \gamma^{ab} K_{ab}^0|_P = x \square_G x \end{aligned} \quad (4.12)$$

measures the deviation from harmonicity of the embedding, which couples via K_{ab}^α to the energy-momentum tensor, and contributes to $\text{Ric}^{ab}[\gamma]$. This is the same mechanism as in [20].

5 Conclusion

In this paper, a formalism for computing the effective curvature of branes in the matrix model is developed. This is done by describing the geometry in terms of an over-complete frame, based on the currents associated with the global $SO(D)$ symmetry of the model. One result is that the effective Ricci tensor has contributions which couple linearly to the energy-momentum tensor. However the coupling is not direct as in general relativity, but somewhat implicit via a coupling tensor \mathcal{P} which depends on the Poisson tensor and the extrinsic curvature of the brane embedding $\mathcal{M} \subset \mathbb{R}^D$. An extra term may lead to vacuum solutions which are not Ricci flat. This mechanism is particularly significant for compactified branes $\mathcal{M} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$, where the coupling \mathcal{P} is always non-vanishing. While the detailed physical consequences depend on the compactification and remain to be clarified, the mechanism clearly leads to a gravity-like long-range force on compactified brane solutions in matrix models, which is not based on the Einstein-Hilbert action. The relation with global symmetries and with non-commutative gauge theory make this mechanism for gravity very attractive for quantization, notably for the maximally supersymmetric IKKT model.

Having confirmed the basic mechanism observed in [20], the tools provided here should allow a more detailed study of the resulting gravity theory. In particular, the additional terms in the geometric equation (3.76) due to \mathcal{P} need to be understood, the response of \mathcal{P} to matter must be clarified, and suitable compactifications must be found. If the resulting gravity turns out to be viable, this would have far-reaching implications. Since target space does not need to be compactified, the vast landscape of 10-dimensional compactifications and its inherent lack of predictivity could be discarded. It suffices instead to consider lower-dimensional brane compactifications of type $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^{10}$, which may also provide the additional structure required for particle physics [26]. Note that there is no contradiction with string theory: the 10-dimensional bulk gravity does indeed arise in a holographic sense. However, bulk gravity is not the dominant mechanism on branes of type $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^{10}$ with B -field, since the present mechanism leads to a 4-dimensional effective gravity, which is clearly dominant for long distances. Note also that in the matrix model there are a priori no propagating degrees of freedom in the bulk, so that we expect no problem with energy leaking off the brane. This is certainly sufficient motivation for more detailed studies.

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Appendix A: Conserved currents

We want to derive the conservation law corresponding to the $SO(D)$ symmetry, which acts as

$$\delta X^A = \lambda_B^A X^B \quad (\text{A.1})$$

for some $\lambda \in \mathfrak{so}(D)$. Consider the corresponding "local" transformation

$$\delta_\epsilon X^A = \frac{1}{2} \lambda_B^A \{ \epsilon(X), X^B \}. \quad (\text{A.2})$$

The corresponding variation of the action is

$$\begin{aligned} \delta S &= Tr \delta X_A \square X^A = \frac{1}{2} Tr \lambda_{AB} \{ \epsilon(X), X^B \} \square X^A \\ &= \frac{1}{2} Tr \epsilon(X) \lambda_{AB} \{ X^B, [X_C, [X^C, X^A]] \}. \end{aligned} \quad (\text{A.3})$$

Using the identity

$$\{A, [B, C]\} = [B, \{A, C\}] - \{C, [B, A]\} \quad (\text{A.4})$$

this becomes

$$\begin{aligned} \delta S &= \frac{1}{2} Tr \epsilon(X) \lambda_{AB} ([X_C, \{X^B, [X^C, X^A]\}] - \{[X^C, X^A], [X_C, X^B]\}) \\ &= \frac{1}{2} Tr \epsilon(X) \lambda_{AB} ([X_C, \{X^B, [X^C, X^A]\}]) \end{aligned} \quad (\text{A.5})$$

as the second term vanishes identically; this reflects the invariance under rigid transformations. This vanishes on-shell for any $\epsilon(X)$, and we obtain the conservation law

$$\begin{aligned} [X_A, \tilde{J}^A] &= 0, \quad \tilde{J}^C = \frac{1}{2} \{ \lambda_{AB} X^A, [X^C, X^B] \} \sim i \theta^{ab} \partial_a X^C J_b, \\ J_b &= \lambda_{AB} x^A \partial_b x^B. \end{aligned} \quad (\text{A.6})$$

This can also be verified directly using the equations of motion (2.5). Note that \tilde{J}^A is a tangential vector field on $\mathcal{M} \subset \mathbb{R}^{10}$. In the semi-classical limit, the conservation law amounts to

$$\begin{aligned} 0 &= \theta^{bc} \partial_b X_A \partial_c (\theta^{ae} \partial_a X^A J_e) = \theta^{bc} \partial_c (g_{ab} \theta^{ae} J_e) \\ &= \gamma^{ce} \partial_c J_e + \rho^{-1} \partial_c (\rho \gamma^{ce}) J_e = e^\sigma (G^{ce} \partial_c J_e - \Gamma^e[G]) J_e \\ &= e^\sigma \nabla^e [G] J_e \end{aligned} \quad (\text{A.7})$$

using the identity (3.88) for $\rho = \sqrt{|\theta^{-1}|}$, and recalling that

$$-\Gamma^a[G] = \frac{1}{\sqrt{|G|}} \partial_b (\sqrt{|G|} G^{ab}) = \rho^{-1} e^{-\sigma} \partial_b (\rho \gamma^{ab}) . \quad (\text{A.8})$$

This is the usual covariant conservation law, once again confirming G as the relevant metric. In Darboux coordinates, this conservation law reduces to

$$e^\sigma \nabla^c [G] J_c \equiv \partial_c (\gamma^{ce} J_e) = 0 . \quad (\text{A.9})$$

Appendix B: Currents and structure constants

We observe the following identity for $\mathfrak{so}(D)$ (resp. $\mathfrak{so}(1, D-1)$) generators

$$f_{\beta\gamma}^\alpha \lambda_{AB}^\beta \lambda_{CD}^\gamma = 2\eta_{BC} \lambda_{AD}^\alpha + 2\eta_{AD} \lambda_{BC}^\alpha - 2\eta_{BD} \lambda_{AC}^\alpha - 2\eta_{AC} \lambda_{BD}^\alpha \quad (\text{B.1})$$

where $f_{\beta\gamma}^\alpha$ are the structure constants of $\mathfrak{so}(D)$. This can be established using the basis $\lambda^\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For the currents J^α , this implies

$$\begin{aligned} f_{\beta\gamma}^\alpha J_b^\beta J_c^\gamma &= f_{\beta\gamma}^\alpha (x^A \lambda_{AB}^\beta \partial_b x^B) (x^C \lambda_{CD}^\gamma \partial_c x^D) \\ &= \partial_b r^2 x^A \lambda_{AD}^\alpha \partial_c x^D + \partial_c r^2 x^C \lambda_{BC}^\alpha \partial_b x^B - 2r^2 \partial_b x^B \lambda_{BD}^\alpha \partial_c x^D \\ &= \partial_b r^2 J_c^\alpha - \partial_c r^2 J_b^\alpha - 2r^2 T_{bc}^\alpha \quad (\alpha \neq 0) \end{aligned} \quad (\text{B.2})$$

and therefore

$$\begin{aligned} r^{-2} f_{\alpha\beta\gamma} J_a^\alpha J_b^\beta J_c^\gamma &= \partial_b r^2 g_{ac} - \partial_c r^2 g_{ab} - 2 \sum_{\alpha \neq 0} J_a^\alpha T_{bc}^\beta \kappa_{\alpha\beta} \\ &= 2r (\partial_b r g_{ac} - \partial_c r g_{ab} - \partial_b r g_{ac} + \partial_c r g_{ab} - \partial_a r g_{bc}) + 2r \partial_a r g_{bc} \\ &= 0 \\ f_{\beta\gamma}^\alpha J^\beta J^\gamma &= 2dr^2 J^\alpha - 2r^2 dJ^\alpha . \end{aligned} \quad (\text{B.3})$$

The first identity can also be seen in NEC, and the last identity also holds for $\alpha = 0$, where both sides vanish. Combining these, (B.3) implies

$$d\theta^\alpha P^{\alpha'\beta} \kappa_{\alpha\alpha'} = \frac{dr}{r} \theta^\beta = \frac{1}{r} \theta^0 \theta^\beta \quad (\text{B.4})$$

which using $\theta^\alpha \theta^\beta \kappa_{\alpha\beta} = 0$ gives (3.14).

Appendix C: Radial curvature

We compute the curvature contribution due to ω . After a suitable translation (or working in NEC) all first-order derivatives ∂r such as ω can be dropped, but we must keep the second derivatives:

$$\begin{aligned} d(r\omega^{\alpha\beta}) &= d\theta^\alpha P^{(r)\beta} - \theta^\alpha dP^{(r)\beta} - dP^{(r)\alpha} \theta^\beta - P^{(r)\alpha} d\theta^\beta \\ r(Pd\omega P)^{\alpha\beta} &= (Pd\theta)^\alpha P^{(r)\beta} - \theta^\alpha (PdP)^{\beta(r)} - (PdP)^{\alpha(r)} \theta^\beta - P^{(r)\alpha} (Pd\theta)^\beta \\ &= -\theta^\alpha (PdP)^{\beta(r)} - (PdP)^{\alpha(r)} \theta^\beta \end{aligned} \quad (\text{C.1})$$

since $Pd\theta = -\theta\omega$ can be dropped. Using $P^{(r)\alpha} = \partial_a r g^{ab} \theta_b^\alpha$ we get

$$(PdP)^{\beta(r)} = \partial_f \partial_d r g^{db} \theta_b^\beta dx^f. \quad (C.2)$$

Therefore

$$\begin{aligned} R[\omega]_{ac} &= \theta_a(d\omega + \omega\omega)\theta_c = \theta_a d\omega \theta_c \\ &= -\theta_a \theta_e^\alpha r^{-1} \nabla_f \partial_d r g^{db} \theta_b^\beta \theta_c dx^e dx^f - \theta_a r^{-1} \nabla_f \partial_d r g^{db} \theta_b^\alpha \theta_c \theta_e^\beta dx^f dx^e \\ &= r^{-2} \left(-g_{ae} \nabla_f J_c^0 + g_{ce} \nabla_f J_a^0 \right) dx^e dx^f \\ &= R[\omega]_{acef} dx^e dx^f \end{aligned} \quad (C.3)$$

where $\nabla = \nabla[g]$, using (3.35) and recalling that $\nabla_a J_b^0 = \frac{1}{2} \nabla_a \partial_b r^2$. Furthermore, we need the contraction

$$r^2 g^{ae} \mathcal{R}_{ac;ef}[\omega] = 2(1-n) \nabla_f J_c^0 - g_{cf} (g^{ae} \nabla_e J_a^0). \quad (C.4)$$

Appendix D: Curvature tensor for special geometries

We note the following identities

$$\begin{aligned} Pd\Lambda^{-1}P_N &= P(\mathbb{1} - \Lambda^{-1})dP_N \\ P_N d\Lambda P &= dP_N(\mathbb{1} - \Lambda)P \end{aligned} \quad (D.1)$$

which follow from $\Lambda P_N = P_N$. Using this and assuming the condition $\nabla Q = 0$ such that $B_\beta^{(\alpha)} = 0$, we obtain using (3.57)

$$\begin{aligned} \Theta_a A A \Theta_b^\dagger &= \Theta_a \Lambda^{-1} d\Lambda P \Lambda^{-1} d\Lambda \Theta_b^\dagger = -\Theta_a d\Lambda^{-1} P d\Lambda \Theta_b^\dagger \\ &= -\Theta_a d\Lambda^{-1} d\Lambda \Theta_b^\dagger + \Theta_a d\Lambda^{-1} P_N d\Lambda \Theta_b^\dagger \\ &= -\Theta_a d\Lambda^{-1} d\Lambda \Theta_b^\dagger + \Theta_a (\mathbb{1} - \Lambda^{-1}) dP_N dP_N (\mathbb{1} - \Lambda) \Theta_b^\dagger. \end{aligned} \quad (D.2)$$

To evaluate

$$\Theta_a dA \Theta_b^\dagger = \Theta_a (dP \Lambda^{-1} d\Lambda P + P d\Lambda^{-1} d\Lambda P - P \Lambda^{-1} d\Lambda dP) \Theta_b^\dagger \quad (D.3)$$

we observe that $PdP_N P = 0$, which implies the following useful identity

$$\begin{aligned} \theta dP &= -\theta dP_N = -\theta dP_N P_N \\ dP \theta^\dagger &= -dP_N \theta^\dagger = -P_N dP_N \theta^\dagger. \end{aligned} \quad (D.4)$$

This gives

$$-\Theta_a P \Lambda^{-1} d\Lambda dP \Theta_b^\dagger = \Theta_a d\Lambda P_N dP_N \Theta_b^\dagger = \Theta_a (\mathbb{1} - \Lambda) dP_N dP_N \Theta_b^\dagger \quad (D.5)$$

as well as

$$\Theta_a dP \Lambda^{-1} d\Lambda P \Theta_b^\dagger = -\Theta_a dP_N P_N d\Lambda \Theta_b^\dagger = -\Theta_a dP_N dP_N (\mathbb{1} - \Lambda) \Theta_b^\dagger. \quad (D.6)$$

Then the metric curvature tensor is obtained using (3.53)

$$\begin{aligned}
R_{ab}[\gamma] &= \theta_a \Lambda dP_N dP_N \Lambda^\dagger \theta_b^\dagger + \Theta_a (dA + AA) \Theta_b^\dagger \\
&= \Theta_a dP_N dP_N \Theta_b^\dagger + \Theta_a (\mathbb{1} - \Lambda^{-1}) dP_N dP_N (\mathbb{1} - \Lambda) \Theta_b^\dagger \\
&\quad + \theta_a (\mathbb{1} - \Lambda) dP_N dP_N \Theta_b^\dagger - \Theta_a dP_N dP_N (\mathbb{1} - \Lambda) \Theta_b^\dagger + \Theta_a \Lambda^{-1} d\omega \Lambda \Theta_b^\dagger \\
&= \theta_a (dP_N dP_N + d\omega) \Lambda \Theta_b^\dagger \\
&= R_{ab}[g] + \theta_a dP_N dP_N Q \theta_b^\dagger
\end{aligned} \tag{D.7}$$

recalling that $\Lambda \Lambda^\dagger = Q + \mathbb{1}$, and dropping $\omega \sim \partial r$ after a suitable translation (or in NEC).

Appendix E: Covariance of conservation laws

Consider

$$\begin{aligned}
G^{ca}(x) \nabla_c [G] T_{ab} &= G^{ca}(x) (\partial_c T_{ab} - \Gamma_{ca}^d T_{db} - \Gamma_{cb}^d T_{ad}) \\
&= G^{ca} \partial_c T_{ab} - \Gamma_{ca}^d T_{db} - G^{ca} \Gamma_{cb}^d T_{ad}
\end{aligned} \tag{E.1}$$

where

$$\Gamma^c = G^{ab} \Gamma_{ab}^c = -\frac{1}{\sqrt{G}} \partial_d (G^{cd} \sqrt{G}). \tag{E.2}$$

we can write

$$G^{ca} \Gamma_{cb}^d T_{ad} = \frac{1}{2} G^{ca} G^{\rho d} T_{ad} (\partial_c G_{eb} + \partial_b G_{ec} - \partial_e G_{cb}) = \frac{1}{2} T^{ce} \partial_b G_{ec} \tag{E.3}$$

where $T^{ce} = G^{ca} G^{ed} T_{ad}$. Therefore

$$\begin{aligned}
G^{ca}(x) \nabla_c [G] T_{ab} &= G^{ca} \partial_c T_{ab} - \frac{1}{2} T^{c\rho} \partial_b G_{\rho c} - \Gamma^\rho T_{\rho b} \\
&= \frac{1}{\sqrt{G}} \partial_c (G^{ca} \sqrt{G} T_{ab}) + \frac{1}{2} \partial_b G^{ca} T_{ac},
\end{aligned} \tag{E.4}$$

where the rhs is valid for any connection.

Finally, we recall the identity (see (2.51), (2.53) in [24])

$$\{X_B, \mathcal{T}_{\text{geom}}^{AB}\} = e^\sigma \square_G x^B \partial_a x_B \theta^{ae} \partial_e x^A \tag{E.5}$$

and note that

$$\begin{aligned}
e^\sigma \square_G x^b &= \{x^A, \{x_A, x^b\}\} = \theta^{ac} \partial_a (\theta^{bd} g_{dc}) \\
&= \theta^{ac} g_{dc} \nabla_a [g] \theta^{bd} = -\gamma^{da} \nabla_a [g] \theta_{cd}^{-1} \theta^{bc}.
\end{aligned} \tag{E.6}$$

Combining these relations gives

$$\{X_B, \mathcal{T}_{\text{geom}}^{AB}\} = -\gamma^{da} \nabla_a [g] \theta_{bd}^{-1} \gamma^{be} \partial_e x^A. \tag{E.7}$$

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